

Mapping functions and critical behavior of percolation on rectangular domains

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The existence probability E_p and the percolation probability P of bond percolation on rectangular domains with different aspect ratios R are studied via the mapping functions between systems with different aspect ratios. The superscaling behavior of E_p and P for such systems with exponents a and b , respectively, found by Watanabe *et al.* [Phys. Rev. Lett. **93**, 190601 (2004)] can be understood from the lower-order approximation of the mapping functions f_R and g_R for E_p and P , respectively; the exponents a and b can be obtained from numerically determined mapping functions f_R and g_R , respectively.

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I. INTRODUCTION

Universality and scaling in critical systems have attracted much attention in recent decades [1–4]. It has been found that critical systems can be classified into different universality classes so that systems in the same class have the same set of critical exponents [1]. According to the theory of finite-size scaling [2–4], if the dependence of a physical quantity M of a thermodynamic system on a parameter t , which vanishes at the critical point $t=0$, is of the form $M(t) \sim t^a$ near the critical point, then for a finite system of linear dimension L , the corresponding quantity $M(L, t)$ is of the form

$$M(L, t) \sim L^{-ay_t} F(tL^{y_t}), \quad (1)$$

where y_t ($\equiv \nu^{-1}$) is the thermal scaling power, $F(x)$ is the scaling function, and $x \equiv tL^{y_t}$ is the scaling variable. It follows from (1) that the scaled data $M(L, t)L^{ay_t}$ for different values of L and t are described by a single finite-size scaling function (FSSF) $F(x)$. Thus it is important to know general features of the scaling function under various conditions. On the basis of renormalization group arguments, Privman and Fisher [3,4] proposed that systems in the same universality classes can have universal finite-size scaling functions (UFSSFs).

Percolation models [5–13] are ideal systems for studying universal and scaling behaviors near the critical point. The system is defined to be percolated when there is at least one cluster extending from the top to the bottom of the system. The average fraction of lattice sites in the percolating clusters is called the percolation probability [5,6] (also known as percolation strength) and will be denoted by P in the present paper. The probability that at least one percolating cluster exists in the system is called the existence probability [9] or crossing probability [13], and will be denoted by E_p in the present paper [14].

Using a histogram Monte Carlo method [9] and other Monte Carlo methods, Hu and collaborators found that the FSSFs depend sensitively on the boundary conditions [15] and shapes [16] of the lattices and many two-dimensional percolation models (including bond and site percolation models on square, honeycomb, and triangular lattices, continuum percolation of soft disk and hard disks, bond percolation on random lattices, etc.) [17] can have universal FSSFs for their E_p and P , and many three-dimensional percolation models (including site percolation on sc, bcc, and fcc lattices and bond percolation on sc lattices) [18] have universal critical exponents and UFSSFs for their E_p and P .

In the present paper, we will study the bond percolation model on $L_1 \times L_2$ rectangular lattices (domains) [19,20], especially the dependence of E_p and P on the aspect ratio R , which is the ratio of the horizontal length L_1 to the vertical length $L_2 \equiv L$ of the system, i.e., $R = L_1/L_2$. For this system, it is well known that the critical bond density ρ_c (also called the bond occupation probability) is $1/2$, the critical exponent β for the percolation probability P is $5/36$, $y_t = 1/\nu = 3/4$, and the critical exponent for the existence probability E_p is 0 [11,21], which with Eq. (1) imply that the dependence of E_p on the linear dimension L , the bond density ρ , and the aspect ratio R can be written as

$$E_p(L, \rho, R) = F_R(\varepsilon L^{y_t}) \quad (\varepsilon \equiv |\rho - \rho_c|/\rho_c) \quad (2)$$

near the critical point ρ_c [3,11]. The function F_R is universal for various percolation models considered in Ref. [17]. The aspect ratio dependence of the function F_R is not clear; e.g., systems with close values of R should have similar forms of F_R ; however, Eq. (2) does not tell us how similar they are. While the value of the existence probability at the critical point $E_p(\rho_c)$ was obtained by Cardy [20], such critical E_p quickly approaches 1 as the aspect ratio is increased, e.g., it is difficult to distinguish the value for $R=16$ from that for $R=32$. Additionally, the region for the rapid increase of E_p shifts to a smaller value of ρ with larger R and fixed L , which cannot be described quantitatively. Therefore, we need other kinds of approach to study the aspect ratio dependence of the UFSSFs.

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Recently, we found “superscaling” of the existence probability E_p and the percolation probability P for bond percolation on rectangular domains with different aspect ratios [22]: $E_p(L, \rho, R) \sim F(\varepsilon' L^y R^a)$ and $P(L, \rho, R) \sim (L^y R^b)^{-\beta} F(\varepsilon' L^y R^b)$, with new exponents a and b , where $\varepsilon' \equiv (\rho - \rho_c')$ with ρ_c' being the effective critical point; a and b were determined by a fitting procedure to be $a=0.14(1)$ and $b=0.05(1)$. In the present paper, we define mapping functions to connect UFSSFs for percolation on domains with different aspect ratios and find that a and b can be obtained from numerically determined mapping functions f_R and g_R for E_p and P , respectively [23].

II. MAPPING FUNCTION AND FINITE-SIZE SCALING

In order to make the meaning of the mapping function clear, we first introduce the mapping function for the percolation model on square lattices with the same aspect ratio. Consider two different square systems A and B with linear sizes L_A and $L_B (> L_A)$, respectively. The existence probability of the system B at density (also called occupation probability) ρ is denoted by $E_p(L_B, \rho)$. The existence probability is a monotonically increasing function whose value changes from 0 to 1 as the density increases from 0 to 1. Therefore, for every $E_p(L_B, \rho)$ of system B , there is a corresponding value ρ' of system A so that the existence probability of system A at ρ' , denoted as $E_p(L_A, \rho')$, is equal to $E_p(L_B, \rho)$ and we have

$$E_p(L_B, \rho) = E_p(L_A, \rho') \equiv E_p(L_A, f_Q(\rho)), \quad (3)$$

which defines the renormalization group transformation from ρ to ρ' in Ref. [9]. The mapping from ρ to ρ' is represented as the mapping function

$$\rho' = f_{A \rightarrow B}(\rho) \equiv f_Q(\rho). \quad (4)$$

The mapping function depends only on the ratio of the system sizes, $Q = L_B/L_A$, and therefore we denote the mapping function by f_Q . The critical point of the system, ρ_c , can be determined from the fixed point of Eq. (3):

$$E_p(L_B, \rho_c) = E_p(L_A, \rho_c),$$

which is equivalent to

$$f_Q(\rho_c) = \rho_c. \quad (5)$$

For fixed Q , ρ_c of Eq. (5) approaches the exact critical point $1/2$ for bond percolation on the square lattice as $L_A \rightarrow \infty$. Table I of the first reference in [9] shows that ρ_c of Eq. (5) is already very accurate for $L_B = 16-20$ and $L_A = 8-16$.

The expansion of the mapping function at the critical point can be written as

$$\begin{aligned} \rho' &= f_Q(\rho) \\ &= f_Q(\rho_c) + \left. \frac{df_Q}{d\rho} \right|_{\rho_c} (\rho - \rho_c) + \dots \end{aligned} \quad (6)$$

It is well known that the thermal scaling power y_t can be obtained from the equation [9]

$$y_t = \frac{\ln \left. \frac{df_Q}{d\rho} \right|_{\rho_c}}{\ln Q}, \quad (8)$$

which implies that the differential coefficient of first order in Eq. (7) is given by

$$\left. \frac{df_Q}{d\rho} \right|_{\rho_c} = Q^{y_t}. \quad (9)$$

We can also obtain the above relation from the viewpoint of the mapping [24].

Using Eqs. (5), (7), and (9), we can rewrite Eq. (3) as

$$E_p(L_B, \rho) = E_p[L_A, \rho_c + Q^{y_t}(\rho - \rho_c)] = E_p(L_A, \rho_c + \varepsilon L_A^{-y_t}) \quad (10)$$

with the scaling variable $\varepsilon \equiv (\rho - \rho_c) L_B^{y_t}$. Since the system sizes L_A and L_B are arbitrary, we can choose L_A as unity and rewrite L_B as L . Then we have

$$E_p(L, \rho) = E_p(1, \rho_c + \varepsilon) \equiv F(\varepsilon), \quad (11)$$

with the scaling function F . Equation (11) implies that the finite-size scaling theory corresponds to the first-order approximation of the expansion in Eq. (7).

To see the behavior of the mapping function f_Q , we determine f_Q from numerically obtained existence probabilities. The procedures to determine f_Q is as follows. (i) Prepare graphs of the existence probabilities vs density. (ii) Draw a line parallel to the horizontal axis. (iii) Determine the two intersection point ρ and ρ' [see the inset of Fig. 1(a)]. (iv) Plot all the pairs (ρ, ρ') by sweeping from $E_p=0$ to 1, then the function $\rho' = f_Q(\rho)$ is obtained. The calculated mapping functions of the bond percolation on square lattices are shown in Fig. 1(a) with aspect ratio $R=1$ and system sizes $L=64, 128$, and 256 . Free boundary conditions are taken for this and other systems in the present paper. All of these functions have linear forms and have a single intersection point, namely, the critical point. The ratio of the slope of these lines gives the value of the critical exponent $\nu=1/y_t$.

Similar arguments can be applied for the percolation probability P . A mapping function g_Q is defined by

$$L_B^{\beta y_t} P(L_B, \rho, R) = L_A^{\beta y_t} P(L_A, g_Q(\rho), R) \quad (12)$$

with a system size ratio $Q = L_B/L_A$. Here, we defined the mapping function with the additional factors $L_B^{\beta y_t}$ and $L_A^{\beta y_t}$ on the left- and right-hand sides of Eq. (12). With such factors, both sides of Eq. (12) are proportional to the FSSF for P as can be seen from Eq. (1). For E_p , the critical exponent is 0 [21] and we need not add such factors in Eq. (3) to define the mapping function. The functions g_Q are shown in Fig. 1(b). Figure 1 shows that the two kinds of functions f_Q and g_Q are identical near the critical point.

The UFSSFs corresponding to the mapping functions in Fig. 1 are shown in Fig. 2. The goodness of the scaling in these figures come from the fact that the mapping functions are well approximated by a linear function when the systems have identical aspect ratio. For the systems with different aspect ratios, the mapping function for E_p becomes highly

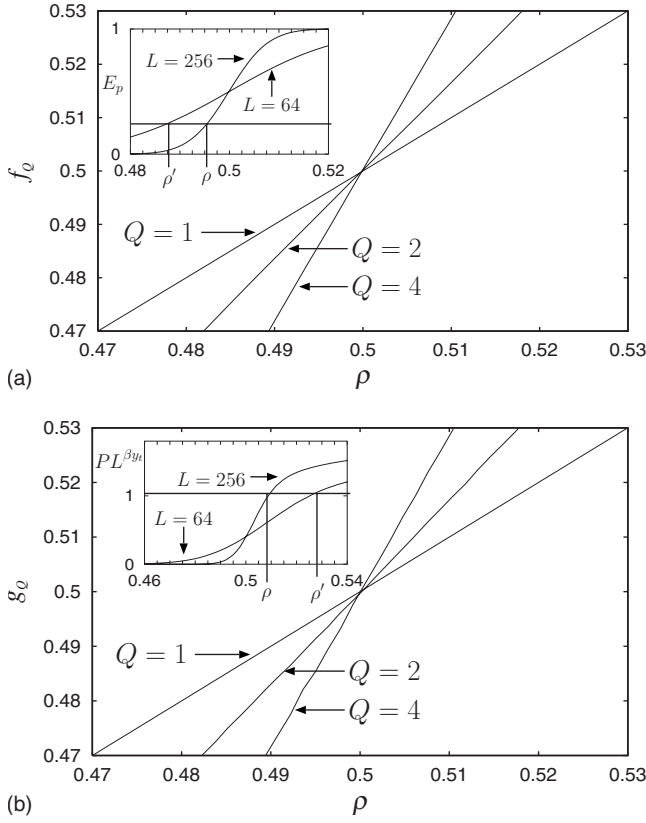


FIG. 1. Mapping functions of the bond percolation model on square lattices with free boundary conditions. (a) f_Q for the existence probabilities E_p as a function of ρ , and (b) g_Q for the percolation probabilities P as a function of ρ . The insets in (a) and (b) show how to construct the mapping functions for E_p and P , respectively.

nonlinear as shown in Fig. 3(b) below. Therefore, the scaling with a low-order approximation does not show good data collapse for large values of the scaling variables (see Fig. 4 in Ref. [22]).

III. MAPPING FUNCTION AND SUPERSCALING

As shown in Fig. 3(a), we define the mapping function $f_R(\rho)$ to map the existence probability of the bond percolation model on an $RL \times L$ lattice, $E_p(L, \rho, R)$, into the existence probability of the bond percolation model on an $L \times L$ lattice, $E_p(L, f_R(\rho), 1)$, with the following equation:

$$E_p(L, \rho, R) = E_p(L, f_R(\rho), 1). \quad (13)$$

The curves of the existence probabilities for the same L and different R do not intersect, since their values at the point ρ_c of the thermodynamic system are different [20]. Therefore, we introduce the effective critical point ρ'_c , which satisfies the equation

$$f_R(\rho'_c) = \rho_c,$$

and consider the expansion of the mapping function f_R at ρ'_c as

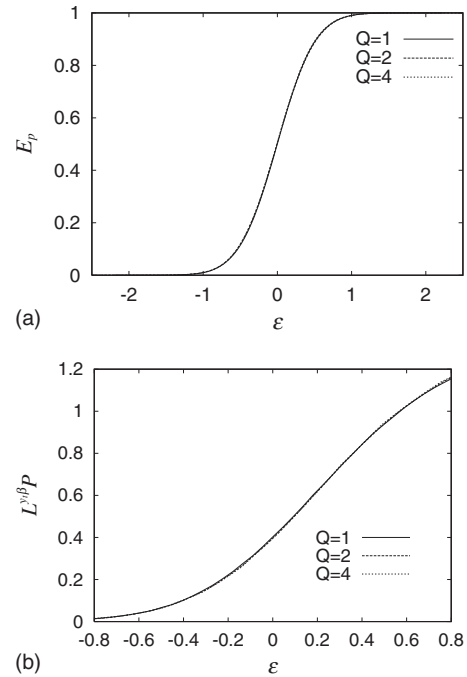


FIG. 2. UFSSFs of (a) existence probabilities and (b) percolation probabilities corresponding to Fig. 1. The aspect ratio is $R=1$, and the system sizes are $L=64$ ($Q=1$), 128 ($Q=2$), and 256 ($Q=4$). The scaling variable is denoted by $\varepsilon \equiv (\rho - \rho_c)L^{1/\nu}$.

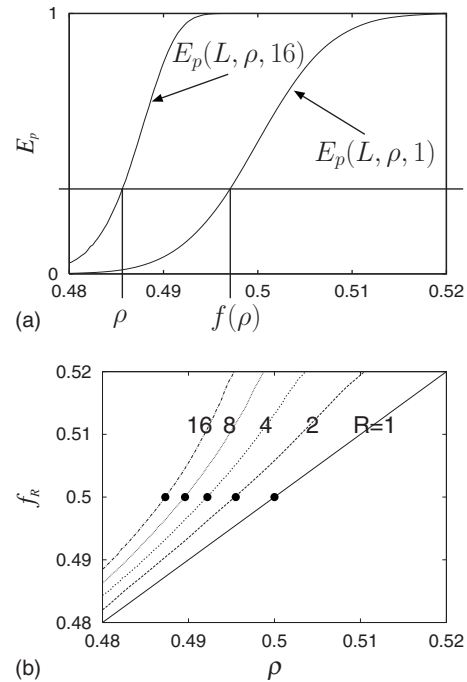


FIG. 3. (a) Procedures to obtain the mapping functions of existence probabilities with different aspect ratios. (b) Obtained mapping functions of systems with $L=256$ and $R=1, 2, 4, 8,$ and 16 (from bottom to top). The solid circles are the effective critical points ρ'_c .

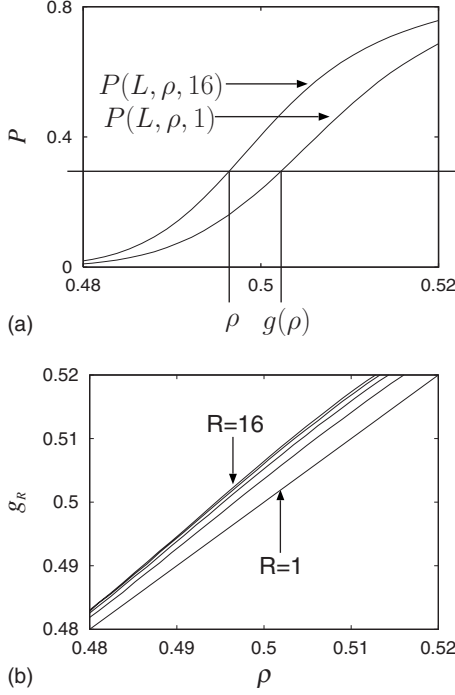


FIG. 4. (a) Procedures to obtain the mapping functions of percolation probabilities with different aspect ratios. (b) Obtained mapping functions of systems with $L=128$ and $R=1, 2, 4, 8,$ and 16 (from bottom to top).

$$f_R(\rho) = f_R(\rho'_c) + \sum_{n=1} \left. \frac{d^n f_R}{d\rho^n} \right|_{\rho'_c} \frac{(\rho - \rho'_c)^n}{n!} \quad (14)$$

$$= \rho_c + \left. \frac{df_R}{d\rho} \right|_{\rho'_c} (\rho - \rho'_c) + \left. \frac{d^2 f_R}{d\rho^2} \right|_{\rho'_c} \frac{(\rho - \rho'_c)^2}{2} + \dots \quad (15)$$

The expansion coefficients depend only on R . With similar arguments in Ref. [24], the aspect ratio dependence of the differential coefficient of the first order can be assumed to be

$$\left. \frac{df_R}{d\rho} \right|_{\rho'_c} \sim R^a, \quad (16)$$

and taking into account up to the first-order derivative of the expansion, we obtain the approximated mapping function as $f_R(\rho) - \rho_c \propto R^a (\rho - \rho'_c)$. From the finite-size scaling,

$$E_p(L, f_R(\rho), 1) \sim F([f_R(\rho) - \rho_c] L^{y_t}), \quad (17)$$

and Eq. (13), we can derive the superscaling form to be

$$E_p(L, \rho, R) = E_p(L, f_R(\rho), 1) \quad (18)$$

$$\sim F([f_R(\rho) - \rho_c] L^{y_t}) \quad (19)$$

$$\sim F((\rho - \rho'_c) L^{y_t} R^a). \quad (20)$$

The mapping functions f_R of the existence probabilities of the bond percolation model on square lattices are shown in Fig. 3(b) for the system size $L=256$ and aspect ratios $R=1,$

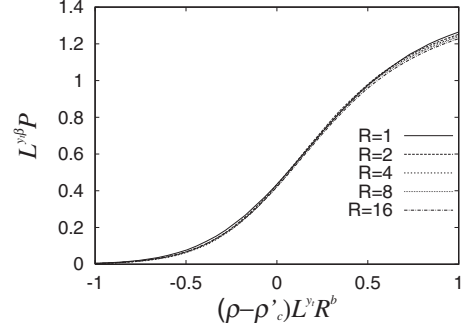


FIG. 5. Superscaling plot of the percolation probabilities P using Eq. (23) for systems with $R=1, 2, 4, 8,$ and $16,$ and $L=128$ with $b=0.05$.

$2, 4, 8,$ and 16 . It shows that these functions do not have any intersection points, since $E_p(L, \rho, R) > E_p(L, \rho, 1)$. Additionally, while the mapping functions of f_Q in Fig. 1 have linear forms, the curves of f_R in Fig. 3(b) for $R > 1$ are not linear.

Consider the mapping of the percolation probabilities with different aspect ratios as

$$P(L, \rho, R) = P(L, g_R(\rho), 1), \quad (21)$$

with a mapping function g_R . Similar to the case of the existence probability, the behavior of the differential coefficient of g_R can be assumed to be

$$\left. \frac{dg_R}{d\rho} \right|_{\rho'_c} \sim R^b, \quad (22)$$

with the effective critical point ρ'_c defined by $g_R(\rho'_c) = \rho_c$ as shown in Fig. 4(a). Note that the value of the effective critical point is different from the one defined by $f_R(\rho'_c) = \rho_c$. We can obtain the superscaling formula as

$$P(L, \rho, R) \sim L^{-\beta y_t} F((\rho - \rho'_c) L^{y_t} R^b). \quad (23)$$

This scaling form is different from the form

$$P(L, \rho, R) \sim (L^{y_t} R^b)^{-\beta} F((\rho - \rho'_c) L^{y_t} R^b) \quad (24)$$

obtained by a heuristic argument in [22]. The mapping functions of percolation probabilities are determined and shown in Fig. 4(b). While they also do not have any intersection points, they have almost linear forms. The scaling plot based on Eq. (23) is shown in Fig. 5. It shows good scaling behavior especially around at the effective critical point. Please note that the right-hand sides of Eqs. (23) and (24) differ only by the factor $R^{-\beta y_t}$ which is very close to 1.

Since the mapping functions f_R for E_p have finite curvature, the scaling formula $E_p(L, \rho, R) \sim F(\varepsilon' L^{y_t} R^a)$ will be improved if we take into account higher orders. This nonlinear scaling form was shown in [22], and it corresponds to an approximate mapping function with a quadratic form, $f_R(\rho) = c_2 \rho^2 + c_1 \rho + c_0$, which is the expression obtained by taking into account up to the second-order derivatives in Eq. (15). With the coefficients $c_2, c_1,$ and c_0 listed in [22], we calculate the differential coefficients and show the results in Fig. 6(a). The mapping functions of the percolation probabilities have an almost linear form; we just fit them with linear function to

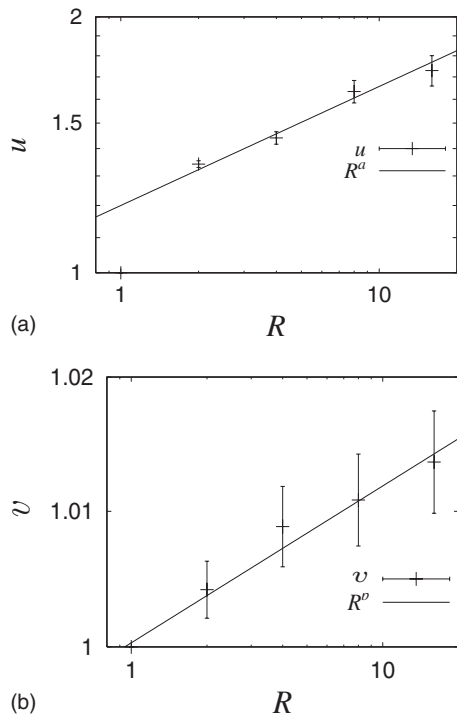


FIG. 6. Plots of (a) $u = df/dp|_{p_c'}$ and (b) $v = dg/dp|_{p_c'}$ as a function R . The solid lines are R^a and R^b with exponents $a=0.14$ and $b=0.05$, respectively. Decimal logarithms are taken for both axes.

obtain the differentiations. The obtained differential coefficients of the percolation probabilities are shown in Fig. 6(b). Figures 6(a) and 6(b) show power law behavior R^a and R^b with values of the exponents $a=0.14$ and $b=0.05$ which are consistent with the results in [22]. Thus a and b can be obtained from the mapping functions f_R and g_R for E_p and P , respectively.

IV. SUMMARY AND DISCUSSION

In summary, we have studied the aspect ratio dependence of FSSFs of the existence and percolation probabilities based on the idea of the mapping functions. From the analysis of

the numerically obtained mapping functions, the superscaling behaviors are found to be the first-order approximation of the mapping functions at the effective critical point. Since the mapping regime used here is pretty general, we can apply it for any other systems and geometric conditions.

Many lattice phase transition models, such as the Ising model, the Potts model, the dilute Potts model, the hydrogen bonding model for water, lattice hard-core particle models, etc., have been shown to correspond to some correlated percolation models [25]. Some of such correlated percolation models have been found to have good finite-size scaling behaviors [26]. It is of interest to use mapping functions to study superscaling in such correlated percolation models and other critical systems [27].

Another interesting development for critical systems is the finite-size corrections for lattice phase transition models [28–33]. Using exact partition functions of the Ising model on finite lattices [34] and exact finite-size corrections for the free energy, the internal energy, and the specific heat of the Ising model [32], Wu, Hu, and Izmailian [35] obtained UFSSFs for the free energy, the internal energy, and the specific heat with analytical equations, which are free of simulation errors. Based on finite-size corrections for the dimer model on finite lattices [36], Izmailian *et al.* [37] obtained UFSSFs for the dimer model with analytical equations, which are also free of simulation errors. It is of interest to extend the above results for the Ising and dimer models to rectangular domains with various aspect ratios. We can then study the superscaling behavior of physical quantities of the Ising and dimer models with the help of mapping functions.

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